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ON RATES OF CONVERGENCE IN THE L2 NORM OF NONPARAMETRIC PROBABI--ETC(U)

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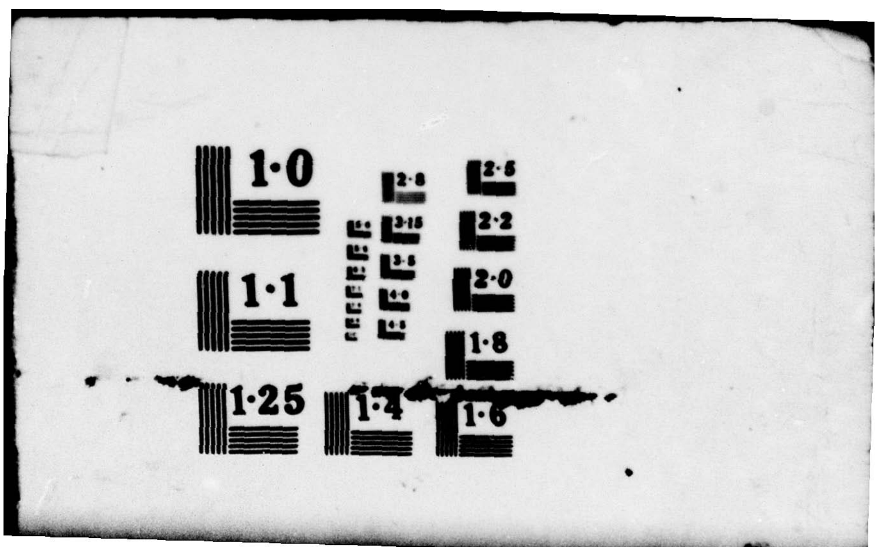
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ON RATES OF CONVERGENCE IN THE L_2-NORM OF NONPARAMETRIC PROBABILITY DENSITY ESTIMATES,

(10) ~~by~~ K. F. Cheng ~~and~~ R. J. Serfling

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ABSTRACT

ON RATES OF CONVERGENCE IN THE L_2-NORM OF NONPARAMETRIC PROBABILITY DENSITY ESTIMATES

For estimation of a probability density function f by an empirical probability density function f_n based on a sample of size n from f, a useful measure of distance is the L_2-norm $||f_n - f|| = (\int[f_n(x) - f(x)]^2dx)^{\frac{1}{2}}$. Considerable study of the rate of *mean square* convergence of $||f_n - f||$ to 0 has taken place. This paper investigates the rate of *almost sure* convergence of $||f_n - f||$ to 0, and characterizes moments $E\{||f_n - f||^k\}$ of order $k > 2$ as well. Application to certain estimation problems in nonprarmetric inference is discussed.

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Key Phrases: Convergence in L_2-norm; Nonparametric density estimation.

1. Introduction. Consider estimation of a probability density function f, defined on the real line, by an empirical probability density function f_n based on a random sample of size n from f. A natural, useful and interesting measure of the distance between f_n and f is given by $||f_n - f|| = (\int[f_n(x) - f(x)]^2 dx)^{\frac{1}{2}}$, $||\cdot||$ denoting the norm of the space $L_2(-\infty, \infty)$. Indeed, considerable attention in the literature (see review papers by Wegman (1972a,b) and Fryer (1977)) has been devoted to the rate of *mean square* convergence of $||f_n - f||$ to 0, that is, the rate of convergence of $E\{||f_n - f||^2\}$ to 0. Further, the *almost sure* (a.s.) convergence of $||f_n - f||$ to 0 has been studied, by Nadaraya (1973), but without investigating the *rate*. This latter question is explored in the present paper. Also, higher moments $E\{||f_n - f||^{2r}\}$ are characterized.

Besides having intrinsic interest, the results have application (see K. F. Cheng and Serfling (1979)) in connection with estimation of efficacy-related parameters such as $T(f) = \int f^2(x)dx$ arising in certain nonparametric inference problems. For this and other such $T(f)$, the approximation of the estimation error $T(f_n) - T(f)$ by the Gâteaux derivative of $T(\cdot)$ at f with increment $f_n - f$ leads to an error term proportional to $||f_n - f||^2$. In order, then, to obtain useful characterizations of the behavior of $T(f_n) - T(f)$, it is necessary that $||f_n - f||^2$ satisfy appropriate conditions. For example, in order ultimately to derive the central limit theorem for $T(f_n) - T(f)$, we require

$$\text{(A)} \qquad n^{\frac{1}{2}}||f_n - f||^2 \to_p 0.$$

To obtain an associated Berry-Esséen rate of order $O(a_n)$, we require a somewhat stronger result of the form

(B) $$P(n^{\frac{1}{2}}||f_n - f||^2 > a_n) = O(a_n),$$

where $a_n \to 0$. Finally, to characterize the a.s. order of magnitude of the fluctations of $T(f_n) - T(f)$, we require

(C) $$n^{\frac{1}{2}}||f_n - f||^2 =_{a.s.} o(g(n)),$$

where $g(n)$ is a function, such as $(\log n)^{\frac{1}{2}}$ or $(\log\log n)^{\frac{1}{2}}$, tending to ∞.

We shall obtain properties (A), (B), (C) as consequences of bounds on the moments $E\{||f_n - f||^{2r}\}$. Theorems 1 and 3 provide suitable bounds under different types of assumptions on f and f_n, and Theorems 2 and 4 provide corresponding implications germane to (A), (B) and (C). We also obtain a further result apropos to (C) (and (A)) by a different approach, utilizing the connection between the L_2-norm and the sup-norm (Theorem 5).

We confine attention to estimators f_n of the *kernal* or *window* type:

$$f_n(x) = (nc_n)^{-1} \sum_{i=1}^{n} K((x - X_i)/c_n), \quad -\infty < x < \infty,$$

where $K(u)$ is a "kernel" (sometimes a probability density) and $\{c_n\}$ is a "bandwidth" sequence of positive constants tending to 0.

The results of this paper admit in some instances extensions to higher dimensional data and analogues for other types of nonparametric density estimator f_n.

2. <u>Properties of $||f_n - f||$</u>. We shall develop order bounds for the moments $E\{||f_n - f||^{2r}\}$, for r a positive integer, by the following approach. Using the elementary inequality $|a + b|^k \le 2^{k-1}(|a|^k + |b|^k)$, for k a positive integer, along with $||f_n - f|| \le ||f_n - Ef_n|| + ||Ef_n - f||$,

we readily obtain

$$(*) \qquad E\{||f_n - f||^{2r}\} \le 2^{2r-1}(E\{||f_n - Ef_n||^{2r}\} + ||Ef_n - f||^{2r}).$$

The right-most term in (*) is simply the r-th power of the integrated squared bias, $\int[Ef_n(x) - f(x)]^2dx$, for which the behavior is known from studies of the mean square error of $||f_n - f||$. See Lemmas 2 and 3 below. The other term on the right-hand side of (*) is an r-th order analogue of the integrated variance. The following result adequately characterizes the behavior of this term for the present purposes.

LEMMA 1. *Let r be a positive integer. Assume that* $\sup_x|K(x)|$ *and* $\int|K(x)|dx$ *are finite. Then*

$$(1) \qquad E\{||f_n - Ef_n||^{2r}\} = O((nc_n)^{-r}), \ n \to \infty.$$

PROOF. Put $Y_{ni}(x) = c_n^{-1}K((x - X_i)/c_n) - c_n^{-1}E\{K((x - X_i)/c_n)\}$. Then

$$||f_n - Ef_n||^2 = \int[n^{-1}\sum_{i=1}^{n} Y_{n,i}(x)]^2dx$$

and thus the left-hand side of (1) may be expressed as

$$(2) \qquad n^{-2r}\sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\cdots\sum_{i_r=1}^{n}\sum_{j_r=1}^{n} E\{\prod_{k=1}^{r}\int Y_{n,i_k}(x_k)Y_{n,j_k}(x_k)dx_k\}.$$

By the assumptions on K, the expectations in (2) are finite and, therefore, by Fubini's theorem,

$$E\{\prod_{k=1}^{r}\int Y_{n,i_k}(x_k)Y_{n,j_k}(x_k)dx_k\} = \int\cdots\int E\{\prod_{k=1}^{r} Y_{n,i_k}(x_k)Y_{n,j_k}(x_k)\}dx_1\ldots dx_k.$$

By independence of the $Y_{n,i}(x)$'s, $1 \le i \le n$, the expectation in the integrand is 0 except in the case that each index in the list $i_1, j_1, \ldots, i_r, j_r$ appears at least twice. In this case the number of distinct elements in the set $\{i_1, j_1, \ldots, i_r, j_r\}$ is $\le r$. It follows that the number of ways to choose $i_1, j_1, \ldots, i_r, j_r$ such that the expectation in (2) is nonzero is $O(n^r)$. Moreover, these nonzero expectations are uniformly $O(c_n^{-r})$. Hence

$$E\{||f_n - Ef_n||^{2r}\} = O(n^{-2r}n^r c_n^{-r}) = O((nc_n)^{-r}). \quad \square$$

The case $r = 1$ has been derived by Nadaraya (1974) with explicit constant in the $O(\cdot)$ expression. Turning now to $||Ef_n - f||$, we first cite a further result of Nadaraya (1974), namely that

$$||Ef_n - f||^2 = O(c_n^{2s}),\ n \to \infty, \tag{3}$$

for f belonging to W_s, K belonging to H_s, and s an even integer ≥ 2. Here W_s denotes the set of functions $f(x)$ having derivations of s-th order inclusively, $s \ge 2$, $f^{(s)}(x)$ being a bounded continuous $L_2(-\infty, \infty)$ function, and H_s, s an even integer, denotes the class of kernels $K(x)$ satisfying

$$K(x) = K(-x),\ \int K(x)dx = 1,\ \sup_x|K(x)| < \infty, \tag{4a}$$

$$\int x^i K(x)dx = 0,\ 1 \le i \le s - 1, \tag{4b}$$

$$\int x^s K(x)dx \ne 0,\ \int x^s|K(x)|dx < \infty. \tag{4c}$$

The possibility of (3) under milder restrictions on the smoothness of f has been investigated by E. P. Cheng and Serfling (1979). Let W_1 denote the set of functions f *either* having bounded continuous derivative

in $L_2(-\infty, \infty)$, *or* being Lipschitz on $(-\infty, \infty)$ and vanishing off a finite interval. Let H_1 denote the class of kernals $K(x)$ satisfying

$$K \text{ nonnegative}, \int K(x)dx = 1, \int x^2 K(x)dx < \infty. \tag{5}$$

For $f \in W_1$ and $K \in H_1$, (3) holds with $s = 1$. Further, it is shown that $||Ef_n - f||^2 = O(c_n^{(2q+1)/(q+1)})$ if f is $\mathrm{Lip}(-\infty, \infty)$ and satisfies the q-th order tail restriction $\int_{|x|>t} f(x)dx = O(t^{-q})$, $t \to \infty$ (implied by $\int |x|^q f(x)dx < \infty$), and if K belongs to H_1. In other words, (3) holds with s given by $(q + \frac{1}{2})/(q + 1)$. Therefore, for $s \in (\frac{1}{2}, 1)$, we define W_s to consist of functions f which are Lip $(-\infty, \infty)$ and satisfy the q-th order tail restriction for $q = (s - \frac{1}{2})/(1 - s)$. Also, for $s \in (\frac{1}{2}, 1)$, we define H_s to be identical with H_1.

Combining all these results on $||Ef_n - f||$, we have

LEMMA 2. *Let* $s \in (\frac{1}{2}, 1)$ *or be an even integer. Let* $f \in W_s$ *and* $K \in H_s$. *Then* (3) *holds.*

Let us now utilize Lemmas 1 and 2 in conjunction with (*). For $c_n = An^{-1/(2s+1)}$ the bounds provided by these lemmas for the terms in (*) have the same order, $O(n^{-2sr/(2s+1)})$. Therefore, we have

THEOREM 1. *Let* $s \in (\frac{1}{2}, 1)$ *or be an even integer, and assume* $f \in W_s$ *and* $K \in H_s$. *Let* r *be a positive integer. Then*

$$E\{||f_n - f||^{2r}\} = O((nc_n)^{-r}) + O(c_n^{2rs}) \tag{6}$$

and, in particular, for $c_n = An^{-1/(2s+1)}$,

$$E\{||f_n - f||^{2r}\} = O(n^{-2sr/(2s+1)}).$$

For the case s an even integer and $r = 1$, relation (6) has been given by Nadaraya (1974) with explicit constants in the $O(\cdot)$ expressions. Those constants were crucial to his purpose of estimating the optimal constant A to be used in the choice $c_n = An^{-1/(2s+1)}$. For our purposes, however, these constants are irrelevant but it is important that r may be chosen arbitrarily large. By virtue of this feature, we are able to establish

THEOREM 2. *Let* $s \in (\frac{1}{2}, 1)$ *or be an even integer. Assume* $f \in W_s$, $K \in H_s$, *and* $c_n = An^{-1/(2s+1)}$. *Then:*

(i) *For* $\alpha < s/(2s + 1)$,

$$n^{\alpha}||f_n - f|| \to_{a.s.} 0. \tag{7}$$

(ii) *For* a_n *given by* $Bn^{-\beta}$, *with* B *constant and* $\beta < (s - \frac{1}{2})/(2s + 1)$,

$$P(n^{\frac{1}{2}}||f_n - f||^2 > a_n) = O(a_n). \tag{8}$$

PROOF. Let $\epsilon > 0$ be given. Let r be a positive integer. Applying Theorem 1, we have

$$\begin{aligned} P(n^{\alpha}||f_n - f|| > \epsilon) &\le \epsilon^{-r}n^{2\alpha r}E\{||f_n - f||^{2r}\} \\ &= O(n^{r[2\alpha - 2s/(2s+1)]}). \end{aligned} \tag{9}$$

For $\alpha < s/(2s + 1)$, (7) thus follows by the Borel-Cantelli lemma and the fact that r may be chosen arbitrarily large. Now replace ϵ by $a_n^{\frac{1}{2}}$ and α by $\frac{1}{2}$ in (9) and obtain

$$P(n^{\frac{1}{2}}||f_n - f||^2 > a_n) \le a_n^{-r}O(n^{r[\frac{1}{2} - 2s/(2s+1)]}). \tag{10}$$

For $a_n = Bn^{-\beta}$, the right-hand side of (10) is seen to be $O(a_n)$ under the condition that

$$\beta < \frac{r}{r+1}\,\frac{s-\frac{1}{2}}{2s+1}.$$

For $\beta < (s - \frac{1}{2})/(2s + 1)$, this condition is satisfied for a sufficiently large choice of r and thus (8) follows. □

REMARK. Regarding the properties discussed in Section 1, we see from Theorem 2(i) that properties (A) and (C) hold for $f \in W_s$ and $K \in H_s$, where s satisfies $s/(2s+1) > \frac{1}{4}$, or equivalently $s > \frac{1}{2}$. That is, (A) and (C) hold for *all possibilities* of f and K covered by the conditions of the theorem. Further, property (B) holds for a_n of the form $Bn^{-\beta}$ with $\beta < (s - \frac{1}{2})/(2s + 1)$; as $s \to \infty$, β may be increased toward $\frac{1}{2}$, which corresponds to the optimal Berry-Esséen rate in typical circumstances. □

Up to now we have considered the behavior of $||f_n - f||$ under a range of smoothness and/or tail restrictions directly imposed on f (and with K satisfying compatible restrictions). Let us now address the question relative to restrictions on the characteristic function ϕ_f of f, following Parzen (1962) and Watson and Leadbetter (1963). The characteristic function ϕ_f is said to *decrease algebraically of degree* p > 0 if

$$0 < \lim_{|t|\to\infty} |t|^p|\phi_f(t)| < \infty.$$

The compatible estimator f_n is of kernal type and, moreover, must be of "*algebriac form*": K must be square integrable and its Fourier transform ϕ_K must be a bounded even square integrable function. As a parallel to Lemma 2, we have

LEMMA 3. *Let* ϕ_f *decrease algebraically of degree* p > 0, *and let* f_n *be of algebraic form with* K *such that* $\int|t|^{-2p}[1 - \phi_K(t)]^2dt < \infty$ *and with* $c_n = Cn^{-1/2p}$. *Then*

(11) $$||Ef_n - f||^2 = O(n^{-(2p-1)/2p}).$$

In fact, Watson and Leadbetter establish (11) for $E\{||f_n - f||^2\}$ and with an explicit constant in the $O(\cdot)$ expression. Their result yields (11) by Jensen's inequality and Fubini's theorem.

Combining Lemmas 1 and 3, we easily obtain

THEOREM 3. *Let* ϕ_f *decrease algebraically of degree* $p > 0$ *and let* f_n *be of algebraic form with* K *bounded, absolutely integrable, and satisfying* $\int |t|^{-2p}[1 - \phi_K(t)]^2 dt < \infty$ *and with* $c_n = Cn^{-1/2p}$. *Let* r *be a positive integer. Then*

$$E\{||f_n - f||^{2r}\} = O(n^{-r(2p-1)/2p}).$$

The preceding result is an analogue to Theorem 1. A corresponding analogue to Theorem 2, by a similar proof as before, is the following.

THEOREM 4. *Assume the conditions of Theorem* 3. *Then:*

(i) *For* $\alpha < (2p - 1)/4p$, (7) *holds.*

(ii) *For* a_n *given by* $Bn^{-\beta}$, *with* $\beta < (p - 1)/2p$, (8) *holds.*

Regarding properties (A), (B) and (C) of Section 1, comments similar to the Remark following Theorem 2 apply. That is, (A) and (C) hold if $p > \frac{1}{2}$; as $p \to \infty$, (B) holds for β arbitrarily close to $\frac{1}{2}$, the optimal value.

We now augment the assertions of Theorems 2(i) and 4(i) with an additional result utilizing the following lemma from Serfling (1979). Here we denote by $||\cdot||_\infty$ the sup-norm, $||h||_\infty = \sup_x |h(x)|$, and by $||\cdot||_1$ the L_1-norm, $||h||_1 = \int |h(x)|dx$.

LEMMA 4. *Let* f *be a probability density on* $(-\infty, \infty)$ *satisfying* $\int_{|x|>t} f(x)dx = O(t^{-q})$, $\to \infty$ *implied by* $\int |x|^q f(x)dx < \infty$). *Let* g_n *be a sequence of probability density functions satisfying* $||g_n - f||_\infty \to 0$. *Then* $||g_n - f||_1 = O(||g_n - f||_\infty^{q/(q+1)})$. *If, further,* f *vanishes off a finite interval, then* $||g_n - f||_1 = O(||f_n - f||_\infty)$.

Various theorems in the literature provide conditions on f and on the estimator f_n (in some cases considering f_n other than the kernel type), under which the rate of a.s. convergence of $||f_n - f||$ to 0 is characterized: for some $\gamma > 0$,

(12) $$n^\gamma ||f_n - f||_\infty \to_{a.s.} 0.$$

In particular, under minimal smoothness conditions on f and for suitable K, results of Reiss (1975), Silverman (1978) and Winter (1978) *establish that* (12) *holds for* $\gamma < 1/3$. *Using such results in conjunction* with Lemma 4 and the inequality $||f_n - f|| \le ||f_n - f||_1 \cdot ||f_n - f||_\infty$, we obtain corresponding results for the L_2-norm. We thus have

THEOREM 5. *Let* f *and* f_n *satisfy conditions under which* (12) *holds for some* $\gamma > 0$. *Let* f *also satisfy the* q*-th order tail restriction of Lemma* 4, *where* $q = \infty$ *denotes that* f *has bounded support. Then, for* $\alpha < \gamma(2q + 1)/2(q + 1)$, (7) *holds.*

Let us briefly compare the rates provided by Theorems 2(i), 4(i) and 5. In Theorem 5 let us take γ approximately $1/3(\gamma < 1/3)$ and q approximately 1 ($q > 1$). Then (7) with α approximately ¼ follows. In this case the assumptions of Theorem 2 are comparable for $s = 3/4$, yielding (7) with α approximately 3/10. Likewise the assumptions of Theorem 4 are comparable for $p = 2$, yielding (7) with α approximately 3/8. Thus Theorem

4 is more effective in this comparison. Furthermore, Theorems 2 and 4 (and their foundations Theorems 1 and 3) provide additional information on $||f_n - f||$. Of course, the approach based on Lemma 4 can also be exploited to provide bounds on $E\{||f_n - f||^{2r}\}$, via

$$E\{||f_n - f||^{2r}\} \leq D_{f,q}\, E\{||f_n - f||_\infty^{r(2q+1)/(q+1)}\},$$

for a constant $D_{f,q}$ depending only on f and q. This requires knowledge of the moments of $||f_n - f||$, a topic which has received little attention in the literature. Leadbetter (1963) has established $E\{||f_n - f||_\infty^2\} = O(n^{-1+1/p-\epsilon})$ under conditions similar to those of Theorems 3 and 4. However, the higher moments of $||f_n - f||_\infty$ have not been investigated.

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20. ABSTRACT

For estimation of a probability density function f by an empirical probability density function f_n sub n based on a sample of size n from f, a useful measure of distance is the L_2-norm, ~~$||f_n - f|| = (\int[f_n(x) - f(x)]^2dx)^{\frac{1}{2}}$~~. Considerable study of the rate of *mean square* convergence of $||f_n - f||$ to 0 has taken place. This paper investigates the rate of *almost sure* convergence of $||f_n - f||$ to 0, and characterizes moments $E\{||f_n - f||^k\}$ of order k > 2 as well. Application to certain estimation problems in nonparametric inference is discussed.

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